A Simple Heat Method for Computing Geodesic Paths on General Manifold Representations


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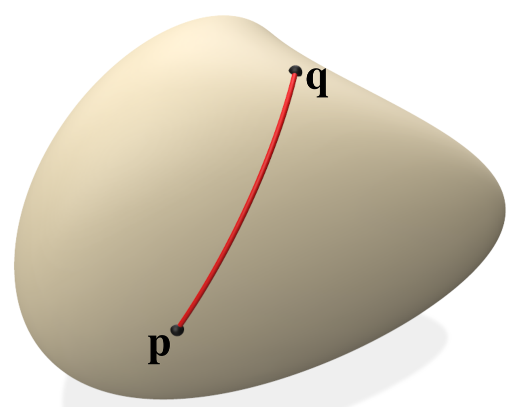
Problem



Geodesic paths are (locally or globally) shortest paths between two points on a manifold. Computation of geodesic paths is a fundamental task in many computer graphics and computer vision applications. We propose a novel algorithm for computing geodesic paths on general manifold representations given only the ability to perform closest point queries.

Most algorithms for geodesic path computation involve minimizing length (e.g., [Yuan et al. 2021]) or geodesic curvature (e.g., [Martinez et al. 2005]). However, as [Yuan et al. 2021] point out, existing methods have mainly been designed specifically for meshes. Instead, we view geodesics in the setting of harmonic maps, which leads to an algorithm that can be applied to meshes, parametric surfaces, point clouds, level sets, exact closest point functions, and more.

Background

A harmonic map $\mathbf{u}(\mathbf{x}) : \mathcal{M} \to \mathcal{N}$ is a mapping from a *source* manifold $\mathcal{M} \subseteq \mathbb{R}^m$ to a *target* $\mathcal{N} \subseteq \mathbb{R}^n$ that minimizes the Dirichlet energy. If $\dim(\mathcal{M}) = 1$, then harmonic maps are geodesics of $\mathcal{N}$.

Method

Our algorithm requires only heat flow on the 1D line segment and the closest point projection $\mathrm{cp}_{\mathcal{N}}$ onto $\mathcal{N}$ — it inherits these attractive features from the method of King and Ruuth [2017]. Given some initial path $\mathbf{u}^0(x) \in \mathcal{N}$, the geodesic path is computed by iterating:

I. Solve $\dfrac{\partial \mathbf{v}}{\partial t} = \dfrac{\partial^2 \mathbf{v}}{\partial x^2}$, $\mathbf{v}(x,0) = \mathbf{u}^k(x)$, for one time step of size Δt using explicit Euler.

II. Project $\mathbf{v}(x, \Delta t)$ onto $\mathcal{N}$ via $\mathbf{u}^{k+1}(x) = \mathrm{cp}_{\mathcal{N}}(\mathbf{v}(x, \Delta t))$.

Discretization

The line segment $\mathcal{M} = [0,1]$ is discretized using equally spaced grid points $x_i = i\Delta x$. The geodesic path $\mathbf{u} \in \mathcal{N}$ is represented discretely as a polyline with vertices $\mathbf{u}_i = \mathbf{u}(x_i)$.

Step I. of the algorithm is applied independently for each of the n dimensions of $\mathbf{u}$. Let u and v denote one of the particular n components of $\mathbf{u}$ and $\mathbf{v}$, respectively. On iteration k, we set $\mathbf{v}_i = \mathbf{u}_i^k$, then apply one step of explicit Euler:

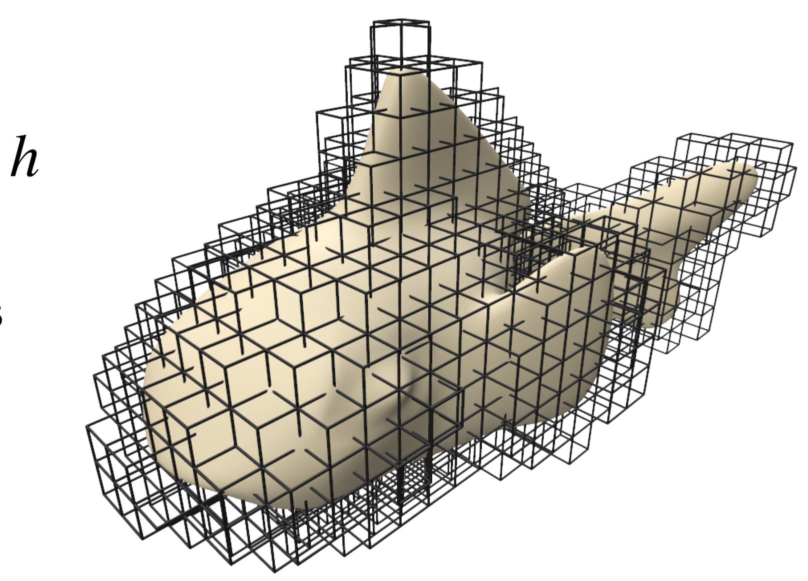
$$v_i^{\Delta t} = v_i + \frac{\Delta t}{\Delta x^2}(v_{i-1} - 2v_i + v_{i+1}).$$

Step II. couples the n dimensions again for each vertex i of the path via $\mathbf{u}_i^{k+1} = \mathrm{cp}_{\mathcal{N}}(\mathbf{v}_i^{\Delta t})$.

Stopping Criteria

We terminate when the average change of the vertices $\dfrac{1}{N+2}\displaystyle\sum_{i=0}^{N+1} \left\| \mathbf{u}_i^{k+1} - \mathbf{u}_i^k \right\| \leq \mathrm{tol}.$

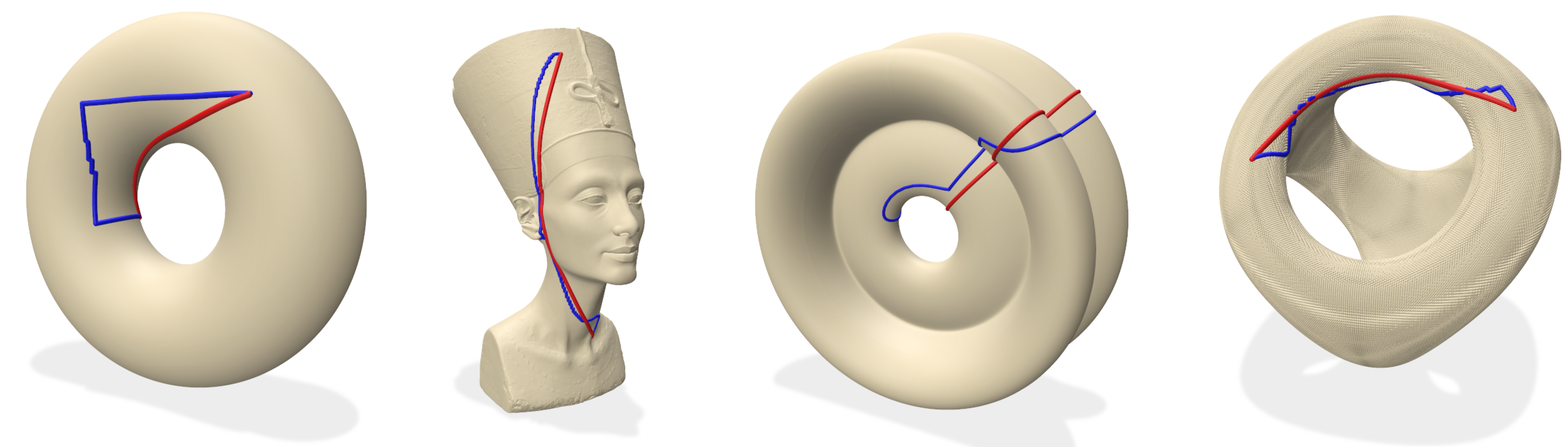
Initial Path Construction



We construct a uniform grid of points $\mathbf{y}_i \in \Omega$ surrounding $\mathcal{N}$ with grid spacing h and $\|\mathbf{y}_i - \mathrm{cp}_{\mathcal{N}}(\mathbf{y}_i)\| \leq 3h/2$. Dijkstra's algorithm is used to compute a path between the nearest grid points in Ω to $\mathbf{p}$ and $\mathbf{q}$. The grid points $\mathbf{y}_i$ in Dijkstra's path are replaced with their previously computed $\mathrm{cp}_{\mathcal{N}}(\mathbf{y}_i)$. Finally, we spatially adapt the initial path, splitting and collapsing edges until all edge midpoints lie within a tolerance to $\mathcal{N}$.

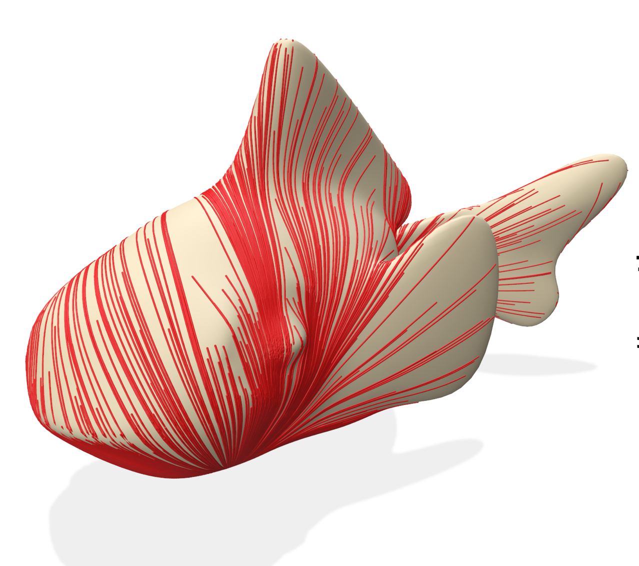
Results

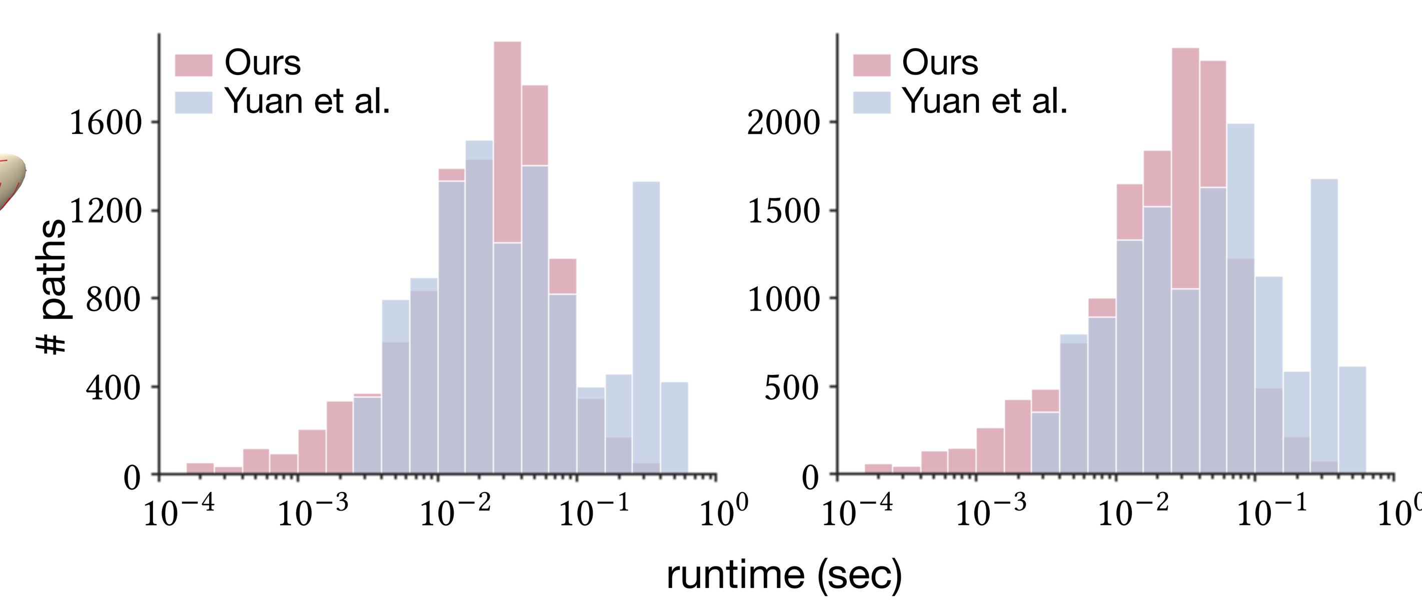
Our algorithm allows computing geodesic paths on any manifold representation that allows closest point queries. Geodesics (red) computed from initial paths (blue) are given on different representations below.

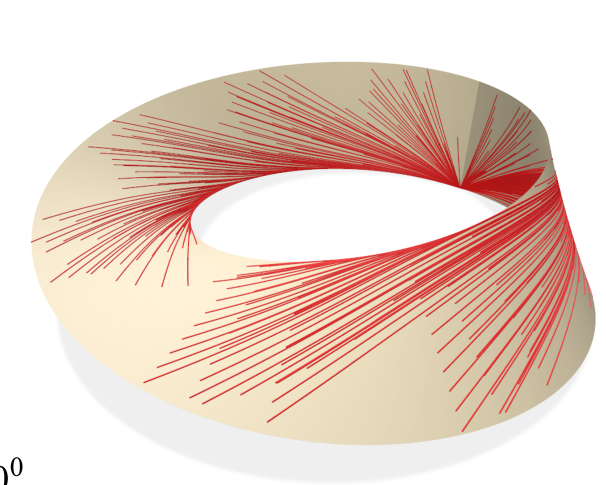


Exact Closest Points | Mesh | Parameterization | Level Set

Our algorithm is faster than the state-of-the-art method for general manifold representations by Yuan et al. [2021].







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