Spatial Adaptivity for Solving PDEs on Manifolds with the Closest Point Method

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Problem

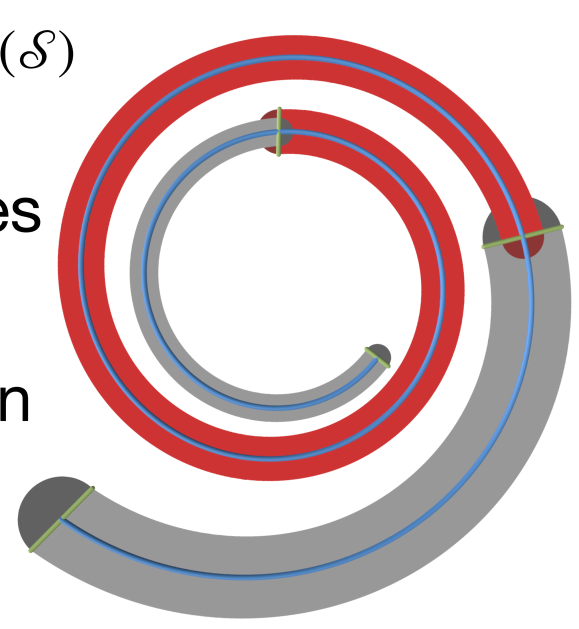
The closest point method (CPM) has been used in for fluid simulations [Morgenroth et al. 2020] and geometry processing [King et al. 2024a, King et al. 2024b]. The most common discretization of CPM uses a uniformly spaced grid, which can be extremely inefficient for manifolds and/or solutions with multiscale features. We propose the first framework to enable adaptivity with CPM, giving an efficient spatial discretization.

Background

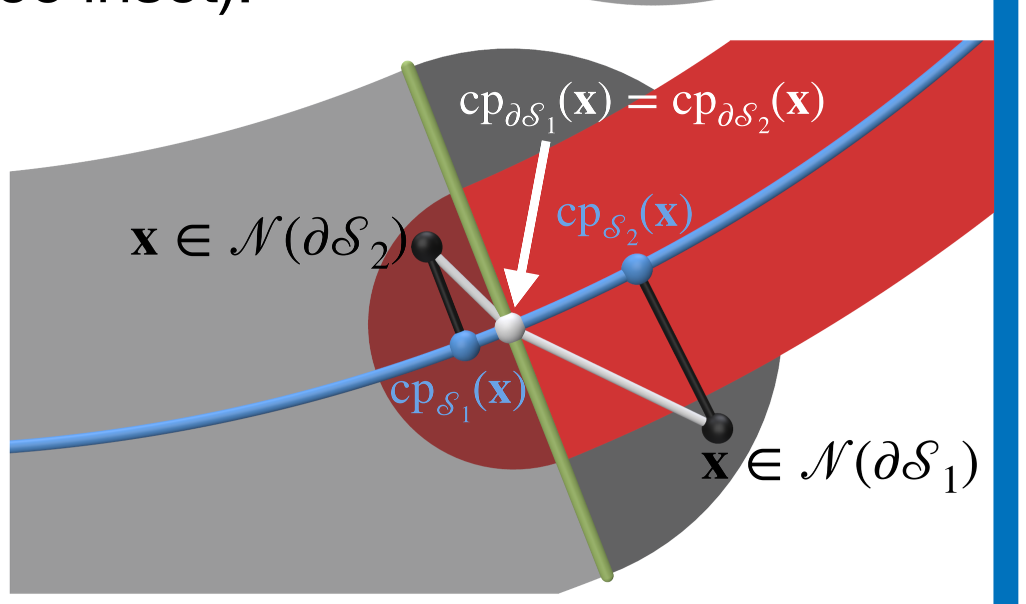
To solve manifold PDEs with CPM, an *embedding PDE* is constructed whose solution agrees with the solution of the manifold PDE at points $\mathbf{y} \in \mathcal{S}$. The embedding PDE is solved on a tubular neighbourhood $\mathcal{N}(\mathcal{S})$ surrounding the manifold $\mathcal{S} \subset \mathbb{R}^d$ with *tube radius* $r_{\mathcal{N}(\mathcal{S})}$.

Method

For spatial adaptivity the tube radius $r_{\mathcal{N}(\mathcal{S})}$ must be allowed to vary over different portions of $\mathcal{S}$. We divide $\mathcal{S}$ into M pieces such that $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_M$. Each subset $\mathcal{S}_m$ is endowed with its own tubular neighbourhood $\mathcal{N}(\mathcal{S}_m)$ with its own tube radius $r_{\mathcal{N}(\mathcal{S}_m)}$ (see inset).



The CP extension operator E is incorrect in the overlap (see inset, darker colours).



Using the CP extension operator from bordering subsets in the overlap provides the correct CP extension for the global problem and couples the subsets. That is, we set

$$E_1 u_{\mathcal{S}}(\mathbf{x}) \equiv E_2 u_{\mathcal{S}}(\mathbf{x}), \quad \text{for} \quad \mathbf{x} \in \mathcal{N}(\partial\mathcal{S}_1),$$

and vice versa for $\mathbf{x} \in \mathcal{N}(\partial\mathcal{S}_2)$.

References

Nathan King, Steven Ruuth, and Christopher Batty. 2024a. A Simple Heat Method for Computing Geodesic Paths on General Manifold Representations. In SIGGRAPH Asia 2024 Posters (SA '24). ACM, New York, NY, USA, Article 69.

Nathan King, Haozhe Su, Mridul Aanjaneya, Steven Ruuth, and Christopher Batty. 2024b. A Closest Point Method for PDEs on Manifolds with Interior Boundary Conditions for Geometry Processing. ACM Transactions on Graphics (2024).

D. Morgenroth, S. Reinhardt, D. Weiskopf, and B. Eberhardt. 2020. Efficient 2D Simulation on Moving 3D Surfaces. Computer Graphics Forum 39, 8 (2020), 27–38.

Results

Figure 1 shows the solution of a Poisson equation on an arc. A fine grid resolution is used only near the centre of the arc where the solution gradient is large.

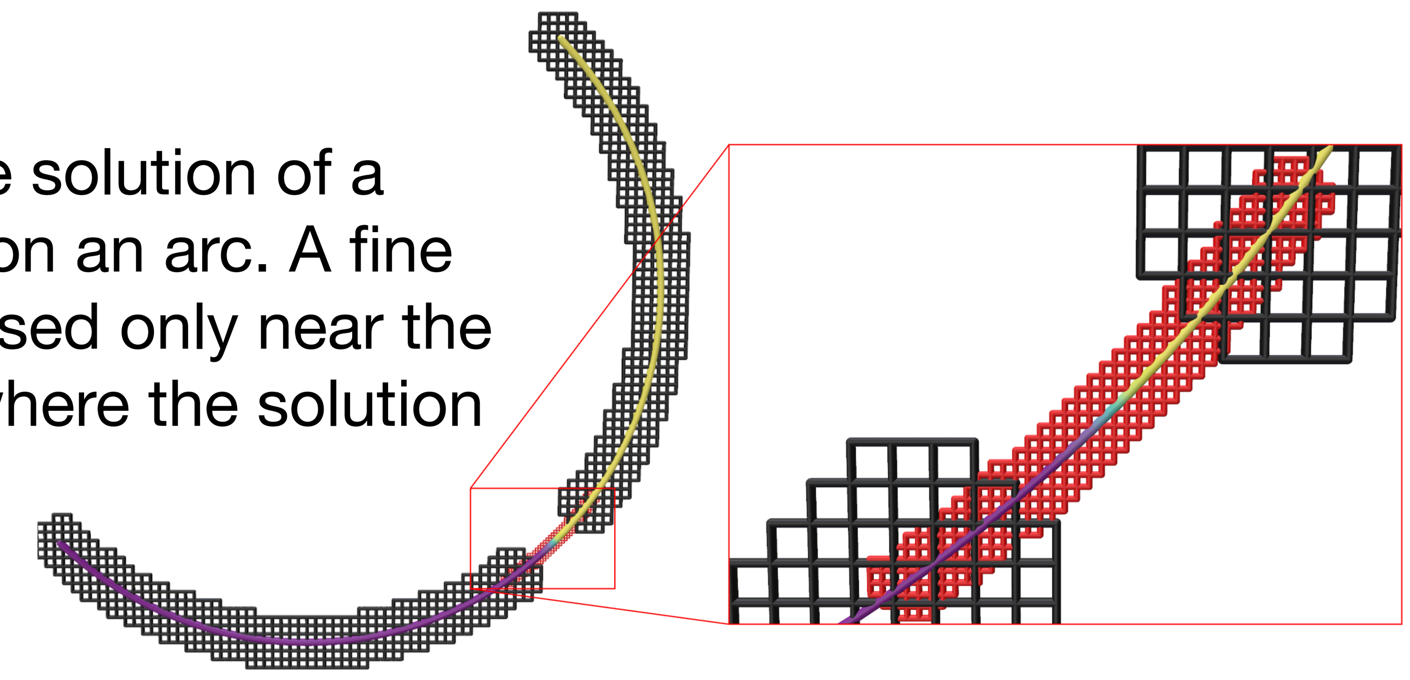


Figure 1: Adaptive computational tube for a Poisson equation on an arc.

Figure 2 shows the behaviour of our adaptive approach as the grid spacing h_2 in the red region varies in comparison to using a uniform computational tube with grid spacing h_2.

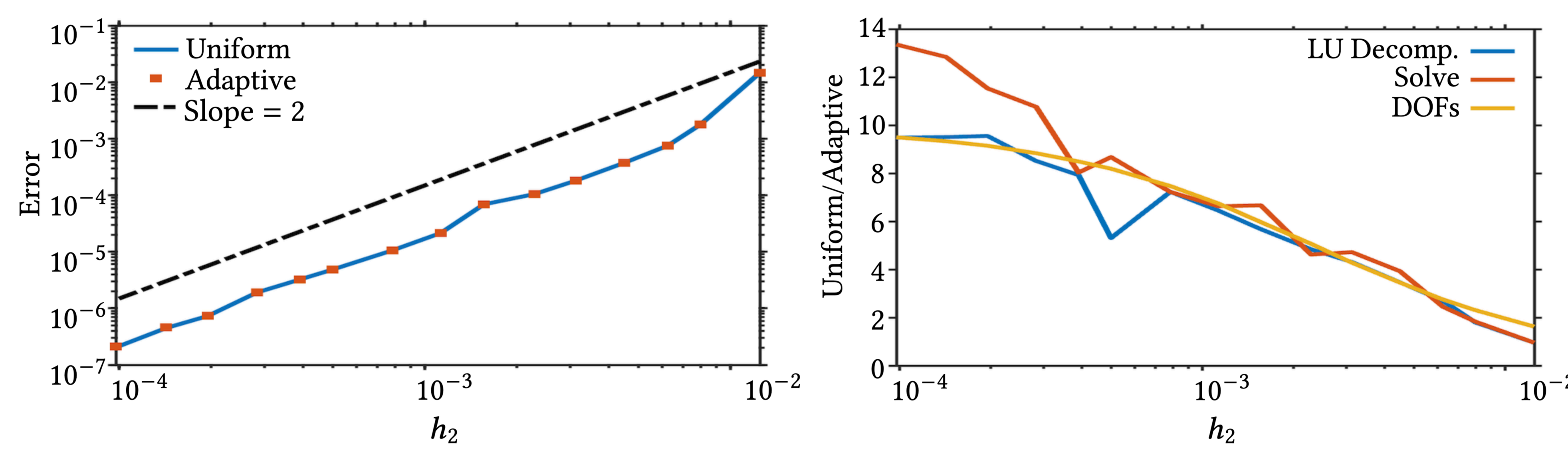


Figure 2: Convergence study (left) and ratio of using a uniform computational tube to the adaptive version for different attributes (right).

A screened-Poisson equation is solved on the spiral sheet shown in Figure 3 (bottom) using an adaptive computational tube (top). A $17\times$ speedup in solve time is achieved with our adaptive tube compared to using a uniform tube with the finest h (see Table 1).

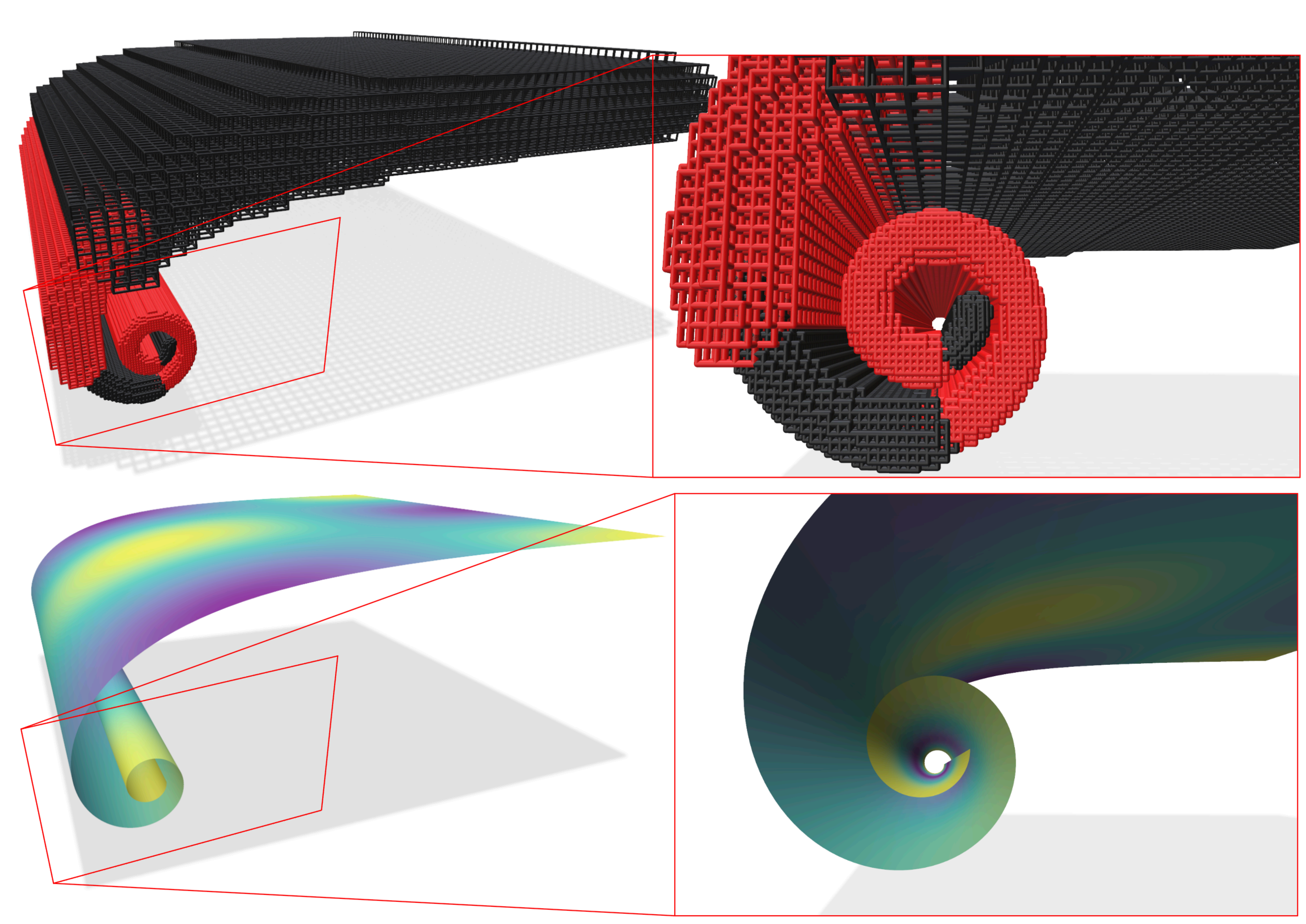


Figure 3: Adaptive computational tube (top) used to compute the solution (bottom) to (4), in the poster abstract, on a spiral sheet.

	$h = 0.064$		$h = 0.008$		Adaptive
DOFs	45,609	$(0.1\times)$	2,862,171	$(6.5\times)$	440,979
Solver Init. Time (s)	2.3×10^{-2}	$(0.08\times)$	3.3	$(11.7\times)$	2.8×10^{-1}
Solve Time (s)	4.2	$(0.06\times)$	1301	$(17.2\times)$	75.5
Solver Iterations	229	$(0.3\times)$	1643	$(1.5\times)$	1094
Max Error	1.6	$(24.6\times)$	9.2×10^{-3}	$(0.14\times)$	6.8×10^{-2}
Average Error	7.4×10^{-1}	$(178\times)$	1.7×10^{-3}	$(0.4\times)$	4.1×10^{-3}

Table 1: Attribute comparison when solving (4), in the poster abstract, on the spiral sheet with uniform computational tubes versus adaptive. The error is computed at 40K equally spaced points in θ and z.